MAA MathFest 2025: Mathematics and Sports

Modeling the Curved Path of a Bowling Ball

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What is the goal of this talk?

This talk describes a SIMIODE modeling scenario,

6-050-BowlingBallPath-ModelingScenario,

that focuses on modeling the path of a bowling ball based on its initial velocity and spin.

- SIMIODE[1] is a community of practice whose goal is to promote the use of modeling in the instruction and learning of differential equations.
- Modeling scenarios are freely available, peer-reviewed pedagogical resources.
  - Modeling scenarios can be explored during class or assigned as group projects.
  - Teacher Versions often include full solutions (accessible with registration)
  - Student Versions exclude solutions and are available to anyone.

[1] `https://qubeshub.org/community/groups/simiode`

Tenpin Bowling

terminology & scoring:

- Bowlers play ten **frames** in which they have up to two opportunities to fell 10 pins.
- A **strike** occurs when all 10 pins are felled in the first attempt.
- A **spare** occurs when any remaining pins are felled in the second attempt.
- The term **pinfall** refers to the number of pins felled on a single attempt.
  - Base Score: Total pinfall for the frame.
  - Strike Bonus: Add pinfall for the next two balls.
  - Spare Bonus: Add pinfall for the next ball.

example: (the maximum possible score is 300)

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<tr><td colspan="2">1</td><td colspan="2">2</td><td colspan="2">3</td><td colspan="2">4</td><td colspan="2">5</td><td colspan="2">6</td><td colspan="2">7</td><td colspan="2">8</td><td colspan="2">9</td><td colspan="3">10</td></tr>
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The Importance of Strikes

Theorem.

If a bowler scores above 190 in a game, there must have been at least one strike.

Proof:

Suppose that the bowler had no strikes. The base pinfall for all ten frames is at most 100. The bonus pinfall for a spare is at most 9, so the bonus pinfall for the game is at most 90. Therefore, the total score is at most 190.

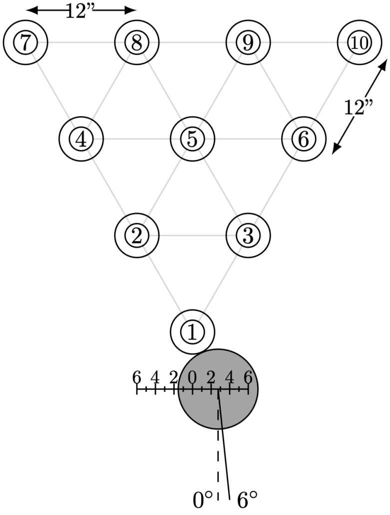
example:

1		2		3		4		5		6		7		8		9		10		
9	1	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9
19		38		57		76		95		114		133		152		171		190		

How to Generate Strikes

The United States Bowling Congress tested the effect of ball placement on the frequency of strikes[2].

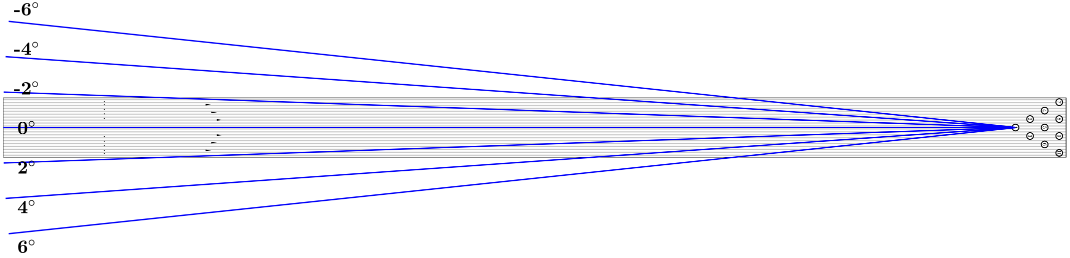
- Tests examined horizontal offset and entry angle (2°, 4°, or 6°) relative to the first pin.
- The observed frequency of strikes exceeded 60% for offsets between 1.5” and 3.25” at all three entry angles.
- The observed frequency of strikes improves to 80% over this range with an entry angle of 6°.
- Ideally, the ball contacts pins 1, 3, 5, and 9.



[2] https://en.wikipedia.org/wiki/Strike_(bowling)

Bowling Lanes are Long and Narrow

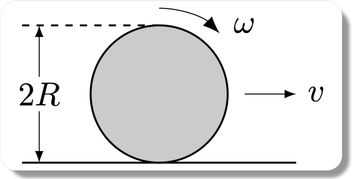
The distance from the foul line to the head pin is 60 feet, while the lane is 3.5 feet wide. The scale diagram below shows the starting location for balls with entry angles between -6° and 6°. Even 2° is impossible with a straight path.



Higher entry angles require the ball to follow a **curved path**.

Rolling with Slipping

Consider a ball of radius R rolling with speed v and angle of rotation ω, as shown below. The velocity at the point of contact is $v - \omega R$.



- $v = \omega R$: The ball is in a state of pure rolling and experiences no acceleration.
- $v > \omega R$: The ball is slipping. Friction acts to decrease v while increasing ω.
- $v < \omega R$: The ball is slipping. Friction acts to increase v while decreasing ω.

Excess spin thus provides the energy required for acceleration, both longitudinally and laterally.

Variables and Parameters

The mathematical model uses a three-dimensional coordinate system obeying the right-hand rule. The origin is located on the foul line at the left side of the lane.

$$
\begin{aligned}
x &\equiv \text{distance from the foul line (feet)} \\
y &\equiv \text{distance from the right-hand gutter (feet)} \\
v_x &\equiv \text{velocity in the } x\text{-direction (feet per second)} \\
v_y &\equiv \text{velocity in the } y\text{-direction (feet per second)} \\
\omega_x &\equiv \text{component of angular velocity in the } x\text{-direction (revolutions per second)} \\
\omega_y &\equiv \text{component of angular velocity in the } y\text{-direction (revolutions per second)} \\
R &\equiv \text{radius of the bowling ball} \\
\mu &\equiv \text{coefficient of friction (varies with location)}
\end{aligned}
$$

The Mathematical Model

One obtains equations of motion for a bowling ball by applying Newton's Second Law in both the linear and angular forms.

The Equations of Motion (assuming $\vec{v}_B \neq 0$)

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = -\mu g \frac{(\vec{v_B})_x}{\|\vec{v}_B\|} \qquad \frac{dv_y}{dt} = -\mu g \frac{(\vec{v_B})_y}{\|\vec{v}_B\|}$$

$$\frac{d\omega_x}{dt} = -\frac{5\mu g}{2R}\frac{(\vec{v_B})_y}{\|\vec{v}_B\|} \qquad \frac{d\omega_y}{dt} = \frac{5\mu g}{2R}\frac{(\vec{v_B})_x}{\|\vec{v}_B\|}.$$

Here, $\vec{v}_B$ denotes the velocity of the ball at the point of contact with the lane, which is given by

$$\vec{v}_B = \vec{v} + \vec{\omega} \times (-R\vec{k}).$$

Oil Patterns

The distribution of oil on a bowling lane is a complicated subject.

- A variety of house and sport patterns are used in different situations.
- House patterns place more oil towards the center of the lane and extend roughly forty feet from the foul line.
- The coefficient of friction is approximately 0.04 in oiled areas and 0.2 in dry areas[3].
- For the purposes of this talk it will be assumed that

$$\mu = \begin{cases} 0.04, & 0 \leq x \leq 40, \\ 0.2, & 40 < x \leq 60. \end{cases}$$

[3] J. Banerjee and J. McPhee, *A Volumetric Contact Model to Study the Effect of Lane Friction and the Radii of Gyration on the "Hook Shot"*, Procedia Engineering, (2014)

Implications of the Model

The model provides insight on the interplay of spin rate, oil distribution, and hook.

- **The magnitude of the frictional force is proportional to the coefficient of friction.**

  This underscores the importance of the dry section at the end of the lane and identifies the sensitivity of hook shots to lane conditions.

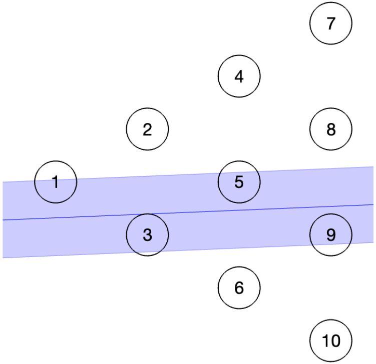
- **The magnitude of the frictional force is independent of the spin rate.**

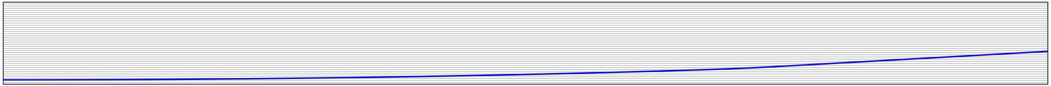
  Excess spin can be thought of like a battery that drains as the ball curves. If the ball achieves pure rolling before it reaches the dry section, the amount of hook will be greatly reduced.

Moderate Spin Rate

A entry angle of about 2.3° can be achieved with the following initial conditions.

$$
\begin{aligned}
x(0) &= 0 & y(0) &= 0.2 \\
v_x(0) &= 20 & v_y(0) &= 0 \\
\omega_x(0) &= -1 & \omega_y(0) &= 2
\end{aligned}
$$



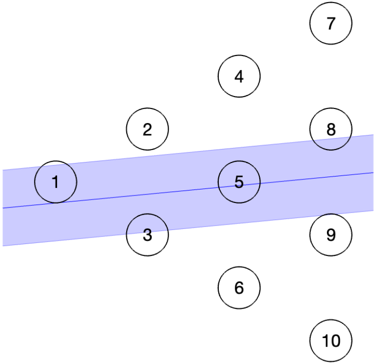


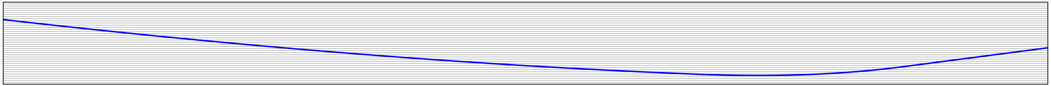
The illustration of the path is scaled 2:1 in the longitudinal direction.

Aggressive Spin Rate

A entry angle of about 5.4° can be achieved with the following initial conditions.

$$x(0) = 0 \qquad y(0) = 2.75$$
$$v_x(0) = 22 \qquad v_y(0) = -1.85$$
$$\omega_x(0) = -4.5 \qquad \omega_y(0) = 1$$





The illustration of the path is scaled 2:1 in the longitudinal direction.

Ideas for Student Projects

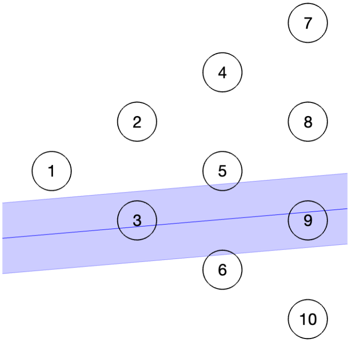
The main objective of the modeling scenario is to guide students as they explore various approaches to increasing the entry angle of a hook shot.

Ideas for Further Exploration:

- Implement more realistic oil patterns according to predefined patterns or that account for heavy lane usage.
- Examine sensitivity of the offset and entry angle to small deviations in initial conditions.
- Investigate the performance of bowling balls with asymmetric cores, designed to produce greater hook.
- Evaluate strategies for common spare patterns based on small changes to the first-ball initial conditions.

Adjusting Speed for Spares

One can adjust the hook for spares by making small changes to the initial velocity v_x.



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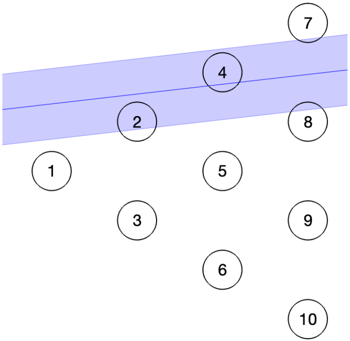
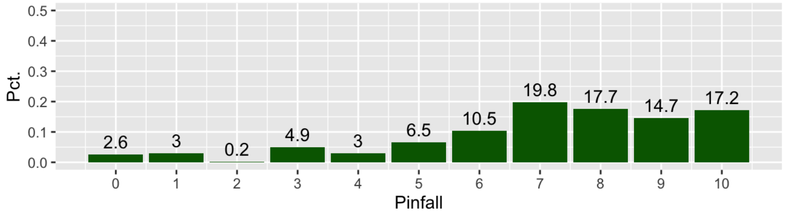


-17%

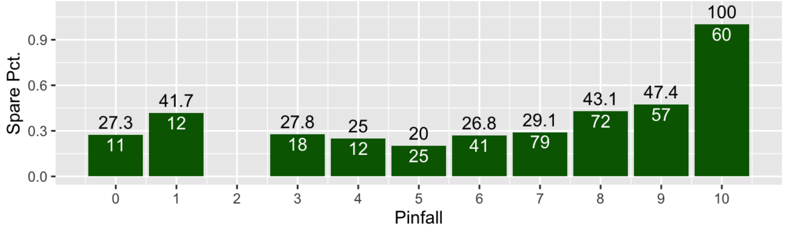
The ten pin can make for a tricky spare with a hook shot. Many bowlers will use a second ball and a straight shot for such spares.

Case Study: Bowling without Spin (≈ 40 games)

First Ball Pinfall:



Spare Completion:



Case Study: Bowling with Spin (≈ 100 games)